

Low- T asymptotic expansion of the solution to the Yang-Yang equation.

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Abstract

We prove that the unique solution to the Yang-Yang equation arising in the context of the thermodynamics of the so-called non-linear Schrödinger model admits a low-temperature expansion to all orders. Our approach provides a rigorous justification, for a certain class of non-linear integral equations, of the low-temperature asymptotic expansion that were argued previously in various works related to the low-temperature behavior of integrable models.

Introduction

The non-linear Schrödinger model refers to a quantum field theory-based model for interacting bosons that is equivalent, in each sector with a fixed number of particles, to a one dimensional gas of bosons interacting through a δ -function potential with strength c . The so-called impenetrable Bose gas which corresponds to $c = +\infty$ was introduced and solved by Girardeau [7]. Then Lieb and Liniger [14] introduced the model at general c and used the so-called coordinate Bethe Ansatz so as to construct eigenfunctions and eigenvalues associated with the underlying spectral problem. The completeness of their construction was however only proven much later by Dorlas [3].

A method for studying the thermodynamics of the model was proposed in the seminal paper of Yang and Yang [16] in 1969. These authors gave an integral representation for the free energy f of the non-linear Schrödinger model

$$f = \int_{\mathbb{R}} \ln [1 + e^{-\frac{\varepsilon(\lambda)}{T}}] \cdot \frac{d\lambda}{2\pi}. \quad (0.1)$$

The above representation is valid for any value $0 < c \leq +\infty$ of the coupling constant just as any value of the chemical potential h . The dependence on the coupling constant and the chemical potential is entirely contained in the function ε which corresponds to the unique solution to the non-linear integral equation

$$\varepsilon(\lambda) = \lambda^2 - h - \frac{T}{2\pi} \int_{\mathbb{R}} K(\lambda - \mu) \ln [1 + e^{-\frac{\varepsilon(\mu)}{T}}] \cdot d\mu \quad \text{with} \quad K(\lambda) = \frac{2c}{\lambda^2 + c^2}. \quad (0.2)$$

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We shall refer to the above equation as the Yang-Yang equation.

The method proposed by Yang and Yang underwent additional developments which allowed its implementation to other integrable models such as the XXZ spin-1/2 spin chain. For the latter model, the thermodynamics were characterized by a solution to a hierarchy of non-linear integral equations [6, 15]. Later, in the nineties, there appeared an alternative method [11, 12] for characterizing the thermodynamics of spin chains. It allowed one to express the free energy of a spin chain in terms of the Trotter limit of the largest eigenvalue of the so-called quantum transfer matrix. In the case of the XYZ model, the latter eigenvalue was shown to be fully characterized in terms of a solution to a single non-linear integral equation [9] quite similar to the Yang-Yang equation (0.2). These further works on the thermodynamics of integrable models clearly point out the Yang-Yang equation as a prototypical form of a non-linear integral equation arising in the study of thermodynamics of integrable models. Even if the resulting representations for the free energies, much in the spirit of (0.1), may appear implicit, they still allow one to obtain the explicit behavior of thermodynamic quantities in the low-temperature regime. For this one first has to compute the low- T asymptotic behavior of the solution to the Yang-Yang equation (or its analogue for other models) and then built on it so as to extract the asymptotic behavior of the integral representation for the free energy such as, eg (0.1). Such calculations were considered by numerous authors and for many different models [2, 5, 9, 10, 13].

The developments mentioned in the previous paragraph were not rigorous. For instance, the method set in '69 by Yang and Yang was formal and has been brought to a satisfactory level of rigor only 20 years later by Dorlas, Lewis and Pulé [4] within the framework of large deviations. Likewise, the works [2, 5, 9, 10, 13] solely proposed consistent algorithms allowing one to compute, order by order, the terms arising in the low- T asymptotic expansion of the solution to non-linear integral equations of Yang-Yang type. However such algorithms always rely on the *a priori* hypothesis that the low-temperature asymptotic expansion exists in the first place and that the remainders remain stable under various operations. In this paper, we address the question of building a rigorous approach to proving the existence of the low-temperature asymptotic expansion of the solution to (0.2). The main result of this paper is:

Theorem 0.1 *Given a positive chemical potential $h > 0$ and a positive coupling constant $c > 0$ for any integer $\ell \in \mathbb{N}$, the unique solution to the Yang-Yang equation admits the low- T asymptotic expansion*

$$\varepsilon(\lambda) = \sum_{k=0}^{\ell} \varepsilon_{2k}(\lambda) T^{2k} + O(T^{2\ell+2}). \quad (0.3)$$

Moreover, for any $0 < a < c/2$, the functions ε_{2k} are even and holomorphic in the open strip \mathcal{S}_a where

$$\mathcal{S}_a = \{z \in \mathbb{C} : |\Im(z)| < a\}. \quad (0.4)$$

Furthermore, the remainder in (0.3) holds uniformly in $\lambda \in \mathcal{S}_a$ and is stable in respect to a finite order λ differentiation.

The paper is organized as follows. In section 1 we review several known properties of the solution $\varepsilon(\lambda)$ to (0.2) and then establish some uniform in T bounds on the location of its zeroes. Then, several properties of integral operators that appear handy in the course of the proof of theorem 0.1 are given in section 2. Finally, in section 3, we prove theorem 0.1.

1 Several properties of the solution to the Yang-Yang equation

We start this section by recalling, for the reader's convenience, Yang-Yang's proof for the existence of solutions to (0.2). We also provide a simple argument so as to justify the uniqueness of solutions. We then push further the

analysis of the objects involved in the proof of the existence of solutions what allows us to provide T -independent bounds for the zeroes of $\varepsilon(\lambda)$. These bounds will play a role in the next section.

Theorem 1.1 [16] *The Yang-Yang equation admits a unique solution $\varepsilon(\lambda)$ such that $[\varepsilon(\lambda) - \lambda^2] \in \mathcal{L}^\infty(\mathbb{R}) \cap \mathcal{O}(\mathbb{C})$. This solution is even, strictly increasing on $]0; +\infty[$ and satisfies to the bounds $\varepsilon'(\lambda) > \lambda$ for any $\lambda \in]0; +\infty[$. Lastly, for $h > 0$, ε admits a unique zero \widehat{q} on $]0; +\infty[$.*

Proof —

The Yang-Yang equation can be recast as the fixed point equation $\mathcal{L}[v](\lambda) = v(\lambda)$ for the functional $\mathcal{L} : L^\infty(\mathbb{R}) \rightarrow L^\infty(\mathbb{R})$ given by

$$\mathcal{L}[f](\lambda) = -h - \frac{T}{2\pi} \int_{\mathbb{R}} K(\lambda - \mu) \ln \left[1 + e^{-\frac{\mu^2 + f(\mu)}{T}} \right] d\mu. \quad (1.1)$$

Indeed, let v correspond to a fixed point of \mathcal{L} , then it is readily seen that $\varepsilon(\lambda) = v(\lambda) + \lambda^2$ solves the Yang-Yang equation. The matter is that \mathcal{L} has at most one fixed point. Indeed, given $f, g \in L^\infty(\mathbb{R})$

$$\left| \mathcal{L}[f](\lambda) - \mathcal{L}[g](\lambda) \right| = \left| \int_{\mathbb{R}} \int_0^1 \frac{K(\lambda - \mu)[g(\mu) - f(\mu)]}{1 + e^{\frac{\mu^2}{T} + \frac{(1-s)}{T}g(\mu) + \frac{s}{T}f(\mu)}} ds \frac{d\mu}{2\pi} \right| \leq \frac{\|f - g\|_{L^\infty(\mathbb{R})}}{1 + \exp \left\{ \frac{1}{T} \max(\|f\|_\infty, \|g\|_\infty) \right\}} \quad (1.2)$$

As a consequence, $\|\mathcal{L}[f] - \mathcal{L}[g]\|_{L^\infty(\mathbb{R})} < \|f - g\|_{L^\infty(\mathbb{R})}$, so that the sharpness of the inequality ensures the uniqueness of a fixed point in $L^\infty(\mathbb{R})$. Note that two distinct solutions to the Yang-Yang equation would give rise to two distinct fixed points of \mathcal{L} . Thus, there is at most one solution to the Yang Yang equation as well.

We now build a sequence converging to the fixed point of \mathcal{L} . Following Yang and Yang the sequence $v_n(\lambda)$ is defined as $v_0(\lambda) = 0$, $v_1(\lambda) = -h$ and $v_{n+1}(\lambda) = \mathcal{L}[v_n](\lambda)$ for $n \geq 1$. The sequence $v_n(\lambda)$ is such that

- $v_n(\lambda)$ is a decreasing sequence in $L^\infty(\mathbb{R})$ of even functions of λ ;
- $v_n(\lambda)$ is smooth and $v'_n(\lambda) \geq 0$ for any $\lambda \in]0; +\infty[$.

These properties are readily seen to hold for $n = 1$. Now, assume that these properties hold for some $n \in \mathbb{N}^*$. Since $v_n \in L^\infty(\mathbb{R})$, one has that $\mathcal{L}[v_n](\lambda)$ is also in $L^\infty(\mathbb{R})$. Derivation under the integral theorems guarantee that $\mathcal{L}[v_n](\lambda)$ is smooth. Since the integrand is negative, it is clear that $v_2 - v_1 < 0$. Else, for $n \geq 2$ one has

$$v_{n+1}(\lambda) - v_n(\lambda) = \int_{\mathbb{R}} K(\lambda - \mu) \left\{ \int_{v_{n-1}(\mu)}^{v_n(\mu)} \frac{ds}{1 + e^{\frac{s + \mu^2}{T}}} \right\} \cdot \frac{d\mu}{2\pi} < 0,$$

and, using that $v'_n(\lambda)$ is odd, one has for $\lambda \in]0; +\infty[$

$$v'_{n+1}(\lambda) = \int_{\mathbb{R}} \frac{K(\lambda - \mu)[2\mu + v'_n(\mu)]}{1 + e^{\frac{v_n(\mu) + \mu^2}{T}}} \cdot \frac{d\mu}{2\pi} = \int_0^{+\infty} [K(\lambda - \mu) - K(\lambda + \mu)] \cdot \frac{2\mu + v'_n(\mu)}{1 + e^{\frac{v_n(\mu) + \mu^2}{T}}} \cdot \frac{d\mu}{2\pi} > 0, \quad (1.3)$$

since, on the one hand $v'_n(\mu) \geq 0$ on $]0; +\infty[$ and on the other hand $K(\lambda - \mu) - K(\lambda + \mu) > 0$ for $(\lambda, \mu) \in (\mathbb{R}_+^*)^2$.

As $\lambda \mapsto v_n(\lambda)$ is increasing, it follows that $v_n(\lambda) \geq v_n(0)$, so that

$$v_{n+1}(0) \geq L(v_n(0)) \quad \text{with} \quad L(x) = -h - T \int_{\mathbb{R}} K(\mu) \ln \left[1 + e^{-\frac{\mu^2}{T} - \frac{x}{T}} \right] \cdot \frac{d\mu}{2\pi}. \quad (1.4)$$

The function

$$L(x) - x = -h - T \int_{\mathbb{R}} K(\mu) \ln \left[e^{\frac{x}{T}} + e^{-\frac{\mu^2}{T}} \right] \cdot \frac{d\mu}{2\pi}. \quad (1.5)$$

is strictly decreasing. The dominated convergence theorem ensures that

$$\lim_{x \rightarrow +\infty} L(x) = -h \quad \text{meaning that} \quad \lim_{x \rightarrow +\infty} [L(x) - x] = -\infty. \quad (1.6)$$

Moreover given any decreasing sequence x_n such that $x_n \rightarrow -\infty$, one has that the increasing sequence of functions $-\ln \left[e^{\frac{x_n}{T}} + e^{-\frac{\mu^2}{T}} \right] K(\mu)$ is bounded from below by the integrable function $-\ln \left[e^{\frac{x_0}{T}} + e^{-\frac{\mu^2}{T}} \right] K(\mu)$. Hence, by the monotone convergence theorem

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}} -K(\mu) \ln \left[e^{\frac{x_n}{T}} + e^{-\frac{\mu^2}{T}} \right] \frac{d\mu}{2\pi} = \int_{\mathbb{R}} \frac{K(\mu) \mu^2}{2\pi T} d\mu = +\infty. \quad (1.7)$$

Thus $\lim_{x \rightarrow +\infty} [L(x) - x] = +\infty$. Since $L(x) - x$ is continuous on \mathbb{R} and strictly decreasing, it admits a unique zero $-z_h$ on \mathbb{R} . Moreover, since $L(0) < -h$ and $L(x) - x$ is strictly decreasing, it follows that $z_h > 0$ for $h > 0$. One can also check that the function L is strictly increasing on \mathbb{R} . Thus,

$$\lim_{x \rightarrow +\infty} L(x) = -h \quad \Rightarrow \quad -h > L(x) \quad \text{for any } x \in \mathbb{R}.$$

As a consequence, one has $-h > L(-z_h) = -z_h$. Then, a straightforward induction shows that $v_n(0) > -z_h$. Since $v_n(\lambda) \geq v_n(0) > -z_h$, one has that $v_n(\lambda)$ is a decreasing sequence of measurable functions that is bounded from below. Thus $v_n(\lambda) \rightarrow v(\lambda)$ pointwise to a measurable function $v(\lambda)$. Taking the pointwise limits in the relations

$$v_n(\lambda) \geq -z_h \quad \text{and} \quad v_n(\lambda) - v_n(-\lambda) = 0 \quad (1.8)$$

ensures that $v(\lambda)$ satisfies to the same properties. As the sequence v_n is decreasing, $0 = v_0(\lambda) \geq v(\lambda)$, and thus $v \in L^\infty(\mathbb{R})$.

It is then enough to apply the monotone convergence theorem so as to get that $v(\lambda)$ corresponds to the fixed point of \mathcal{L} in $L^\infty(\mathbb{R})$. Indeed, for any λ , one has that

$$\mu \mapsto h_n(\mu) = K(\lambda - \mu) \ln \left[1 + e^{-\frac{v_n(\mu) + \mu^2}{T}} \right] \quad (1.9)$$

is an increasing sequence of positive measurable functions which, due to $v_n(\lambda) \geq v_n(0) \geq -z_h$, is bounded from above by a $L^1(\mathbb{R})$ function

$$h_n(\mu) \leq K(0) \ln \left[1 + e^{-\frac{\mu^2 - z_h}{T}} \right]. \quad (1.10)$$

Since h_n converges pointwise, one has that

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}} h_n(\mu) \cdot \frac{d\mu}{2\pi} = \int_{\mathbb{R}} K(\lambda - \mu) \ln \left[1 + e^{-\frac{v(\mu) + \mu^2}{T}} \right] \cdot \frac{d\mu}{2\pi}. \quad (1.11)$$

Taking the pointwise limit in $v_{n+1}(\lambda) = \mathcal{L}[v_n](\lambda)$ shows that v is indeed the fixed point of \mathcal{L} .

Finally, it follows from the integral representation given by $v(\lambda) = \mathcal{L}[v](\lambda)$ that v is holomorphic in the strip S_c . By using this information and deforming the integration contour \mathbb{R} in $\mathcal{L}[v](\lambda)$, it is readily seen that v is

holomorphic in \mathcal{S}_{2c} . Hence, by immediate induction, it is entire. A derivation under the integral sign ensures that $v'(\lambda) > 0$ for $\lambda \in]0; +\infty[$. Hence, setting $\varepsilon(\lambda) = \lambda^2 + v(\lambda)$ we get the sought unique solution to the Yang-Yang equation. Moreover, we immediately have that the latter satisfies $\varepsilon'(\lambda) > 0$ on $[-q; q]$, $\varepsilon(0) < -h$ and $\varepsilon(\lambda) \geq -z_h + \lambda^2$. Thus $\lim_{\lambda \rightarrow +\infty} \varepsilon(\lambda) = +\infty$, so that ε admits a unique zero \widehat{q} on $]0; +\infty[$. Hence, by evenness, it has thus two zeroes $\pm \widehat{q}$ on \mathbb{R} . ■

Proposition 1.1 *Let $h > 0$, then there exists T_0 such that for any $T \in [0; T_0]$ the unique zero \widehat{q} of ε as defined in theorem 1.1 satisfies $\sqrt{h} < \widehat{q} < 2\sqrt{w}$, where $w \in]0; +\infty[$ corresponds to the unique zero of*

$$\omega(x) = x - h - \frac{2(x + c^2)}{\pi} \arctan\left(\frac{\sqrt{x}}{c}\right) + \frac{2c}{\pi} \sqrt{x}. \quad (1.12)$$

Proof — Evaluating the Yang-Yang equation at $\lambda = \widehat{q}$ leads to

$$\widehat{q}^2 = h + \frac{T}{2\pi} \int_{\mathbb{R}} K(\widehat{q} - \mu) \ln[1 + e^{-\frac{\varepsilon(\mu)}{T}}] \cdot d\mu. \quad (1.13)$$

The second term is positive meaning that $\widehat{q} \geq \sqrt{h}$, this uniformly in $T \in [0; +\infty[$.

It thus remains to obtain the upper bound. It was established in the course of the proof of theorem 1.1 that $\varepsilon(\lambda) \geq \lambda^2 - z_h$ where z_h solves $-z_h = L(-z_h)$ with L defined in (1.4). Thus $\widehat{q} \leq \sqrt{z_h}$ and the upper bound will follow from an estimate on z_h . For this purpose, we recast the defining equation for z_h in the form

$$z_h = h + \int_0^{\sqrt{z_h}} (z_h - \mu^2) K(\mu) \cdot \frac{d\mu}{\pi} + V_h(T) \quad \text{where} \quad V_h(T) = \frac{T}{\pi} \int_0^{+\infty} K(\mu) \ln[1 + e^{-\frac{|\mu^2 - z_h|}{T}}] \cdot d\mu \quad (1.14)$$

It is readily seen that $V_h(T) \leq T \ln 2$. Explicit integration of the second terms in (1.14) shows that z_h solves the equation $\omega(z_h) = V_h(T)$. The function ω is smooth on $]0; +\infty[$ and strictly increasing

$$\omega'(x) = 1 - \frac{2}{\pi} \arctan\left(\frac{\sqrt{x}}{c}\right) > 0. \quad (1.15)$$

Since $\omega(0) = -h$ and $\omega(x) \sim 4c\sqrt{x}/\pi$ as $x \rightarrow +\infty$, it follows that ω admits a unique zero w on $]0; +\infty[$. Moreover ω is a C^∞ -diffeomorphism from $]0; +\infty[$ onto $]-h; +\infty[$. Since $V_h(T) > 0$, one has that $V_h(T) \in]-h; +\infty[$ so that $z_h = \omega^{-1}(V_h(T))$. Then, the Taylor expansion of $\omega^{-1}(t)$ to the first order around $t = 0$ and the estimates for $V_h(T)$ ensure that $z_h = w + O(T)$. In particular, as ω^{-1} is strictly increasing, it follows that there exists T_0 such that $w \leq z_h \leq 4w$ on $[0; T_0]$. As a consequence, $\widehat{q} \leq 2\sqrt{w}$ uniformly in $T \in [0; T_0]$. ■

2 Some properties of a linear integral equations driven by the Lieb kernel

2.1 Invertibility of the Lieb operator

Lemma 2.1 *Let $K(\lambda)$ be the Lieb kernel defined in (0.2) and $f \in L^1([-\alpha; \alpha])$, then*

$$\left| \int_{-\alpha}^{\alpha} f(\mu) K(\lambda - \mu) d\mu d\lambda \right| \leq \int_{-\alpha}^{\alpha} |f(\mu)| d\mu \cdot \int_{-\alpha}^{\alpha} K(\lambda) d\lambda, \quad (2.1)$$

Proof —

One has

$$\left| \int_{-\alpha}^{\alpha} f(\mu) K(\lambda - \mu) d\mu d\lambda \right| \leq \int_{-\alpha}^{\alpha} d\mu \left\{ |f(\mu)| \int_{-\alpha-\mu}^{\alpha-\mu} K(\lambda) d\lambda \right\}. \quad (2.2)$$

The integral involving the kernel K can be decomposed as

$$\int_{-\alpha-\mu}^{\alpha-\mu} K(\lambda) \cdot d\lambda = \int_{-\alpha}^{\alpha} K(\lambda) \cdot d\lambda + \int_{-\alpha-\mu}^{-\alpha} K(\lambda) \cdot d\lambda - \int_{\alpha-\mu}^{\alpha} K(\lambda) \cdot d\lambda. \quad (2.3)$$

Yet, for any $\mu \in [-\alpha; \alpha]$, one has that

$$\int_{-\alpha-\mu}^{-\alpha} K(\lambda) \cdot d\lambda - \int_{\alpha-\mu}^{\alpha} K(\lambda) \cdot d\lambda = \int_{\alpha}^{\alpha+\mu} [K(\lambda) - K(\lambda - \mu)] \cdot d\lambda \leq 0. \quad (2.4)$$

This is evident for $\alpha \geq \mu \geq 0$, since then $[K(\lambda) - K(\lambda - \mu)] \leq 0$ for any $\lambda \geq 0$. The case when $-\alpha \leq \mu \leq 0$ follows since the orientation of the integral produces a sign and $[K(\lambda) - K(\lambda - \mu)] = [K(\lambda) - K(\lambda + |\mu|)] \geq 0$ for $\lambda \geq 0$. As, the last two terms in (2.3) produce negative contributions, one gets

$$\int_{-\alpha-\mu}^{\alpha-\mu} K(\lambda) d\lambda \leq \int_{-\alpha}^{\alpha} K(\lambda) d\lambda. \quad (2.5)$$

■

Lemma 2.2 *Let $\alpha \in [0; +\infty[$. The integral operator $K/2\pi$ acting on $L^2([-\alpha; \alpha])$ with the integral kernel $K(\lambda - \mu)$ given in (0.2) is a compact trace class operator. The logarithm of the Fredholm determinant of $I - K/2\pi$ is well defined and given by the convergent series*

$$\ln \det_{[-\alpha; \alpha]} \left[I - \frac{K}{2\pi} \right] = -\frac{2\alpha K(0)}{2\pi} - \sum_{n=1}^{+\infty} \frac{1}{n} \int_{-\alpha}^{\alpha} \prod_{a=1}^n K(\tau_a - \tau_{a+1}) \cdot \frac{d^n \tau}{(2\pi)^n}, \quad (2.6)$$

where in each n -fold integral periodic boundary condition $\tau_{n+1} = \tau_1$ are being undercurrent. Finally, the operator $I - K/2\pi$ is invertible and the kernel $R^{(\alpha)}(\lambda, \mu)$ of the resolvent operator $R^{(\alpha)}$ can be expressed in terms of the below series of multiple integrals

$$R^{(\alpha)}(\lambda, \mu) = K(\lambda - \mu) + \sum_{n=1}^{+\infty} \int_{-\alpha}^{\alpha} K(\lambda - \tau_1) \cdot \prod_{a=1}^{n-1} K(\tau_a - \tau_{a+1}) \cdot K(\tau_n - \mu) \cdot \frac{d^n \tau}{(2\pi)^n}. \quad (2.7)$$

Given any $0 < a < c/2$, the series representation (2.7) is convergent in respect to the sup norm on $\mathcal{S}_a \times \mathcal{S}_a$ (with \mathcal{S}_a as in (0.4)). Moreover, for any $(\lambda, \mu) \in \mathcal{S}_a^2$, with $0 < a < c/2$ one has the estimates

$$|R^{(\alpha)}(\lambda, \mu)| = \|K\|_{L^\infty(\mathcal{S}_{2a})} \cdot \left\{ 1 + \frac{C_K}{1 - \|K/(2\pi)\|_{L^1([-\alpha; \alpha])}} \right\} \quad \text{where} \quad C_K = \sup_{\lambda \in \mathcal{S}_a} \int_{-\alpha}^{\alpha} |K(\lambda - \tau)| \frac{d\tau}{2\pi} \quad (2.8)$$

and the resolvent is an analytic function of (λ, μ) on the strip $\mathcal{S}_{c/2} \times \mathcal{S}_{c/2}$

Finally, the function $(\alpha, \lambda, \mu) \mapsto R^{(\alpha)}(\lambda, \mu)$ is C^1 on \mathbb{R}^3 .

Proof —

The estimates for any of the n -fold multiple integrals occurring in (2.6)-(2.7) can be obtained by using the majoration (2.1). We treat the case of the n -fold cyclic integral occurring in the series expansion for $\ln \det[I - K/2\pi]$:

$$\begin{aligned} \int_{-\alpha}^{\alpha} \prod_{a=1}^n K(\tau_a - \tau_{a+1}) \cdot \frac{d^n \tau}{(2\pi)^n} &\leq K(0) \int_{-\alpha}^{\alpha} K(\tau_1 - \tau_2) \dots K(\tau_{n-1} - \tau_n) \cdot \frac{d^n \tau}{(2\pi)^n} \\ &\leq \frac{K(0)}{(2\pi)^n} \cdot \|K\|_{L^1([-\alpha; \alpha])} \int_{-\alpha}^{\alpha} K(\tau_1 - \tau_2) \dots K(\tau_{n-2} - \tau_{n-1}) \cdot d^{n-1} \tau \leq \frac{2\alpha K(0)}{(2\pi)^n} \|K\|_{L^1([-\alpha; \alpha])}^{n-1} . \end{aligned} \quad (2.9)$$

It is readily checked that

$$\int_{\mathbb{R}} K(\lambda) \frac{d\lambda}{2\pi} = 1 \quad \text{meaning that} \quad \|K\|_{L^1([-\alpha; \alpha])} < 2\pi . \quad (2.10)$$

As a consequence, the series representation for $\ln \det[I - K/2\pi]$ is convergent. The same types of estimates also ensure the uniform convergence in respect to the sup norm topology on compact subsets of \mathbb{R}^2 of the series (2.7) defining the resolvent.

Lastly, the fact that $(\alpha, \lambda, \mu) \mapsto R^{(\alpha)}(\lambda, \mu) \in C^1(\mathbb{R}^3)$ can be readily inferred from the Fredholm series representation for $R^{(\alpha)}(\lambda, \mu)$, $\det_{[-\alpha; \alpha]}[I - K/(2\pi)]$ and the fact that $\det_{[-\alpha; \alpha]}[I - K/(2\pi)] \neq 0$ for any given $\alpha \in [0; +\infty[$. ■

2.2 Positivity property of Lieb-like kernels

Lemma 2.3 *Let $I - O$ be a trace class integral operator on $L^2([-a; a])$ with $0 < a < +\infty$ whose integral kernel is continuous and satisfies*

$$O(\lambda, \mu) - O(\lambda, -\mu) \geq 0 \quad \text{and} \quad O(\lambda, \mu) = O(-\lambda, -\mu) \quad \text{for any} \quad \mu, \lambda \in]0; a[. \quad (2.11)$$

Assume that the series representation

$$R_O(\lambda, \mu) = O(\lambda, \mu) + \sum_{n \geq 1} \int_{-a}^a O(\lambda, \tau_1) \cdot \prod_{\ell=1}^{n-1} \{O(\tau_\ell, \tau_{\ell+1})\} \cdot O(\tau_n, \mu) \cdot d^n \tau \quad (2.12)$$

for the resolvent kernel $R_O(\lambda, \mu)$ of the inverse operator $I + R_O$ to $I - O$ is convergent in $L^\infty([-a; a]^2)$. Then, the resolvent kernel $R_O(\lambda, \mu)$ satisfies

$$R_O(\lambda, \mu) - R_O(\lambda, -\mu) \geq 0 \quad \text{for any} \quad \mu, \lambda \in]0; a[. \quad (2.13)$$

Proof —

We consider the series representation for $R_O(\lambda, \mu) - R_O(\lambda, -\mu)$ and show that every term in this series is positive. Due to the hypothesis of the lemma regarding to the kernel $O(\lambda, \mu)$, it is enough to show that

$$\begin{aligned} \int_{-a}^a O(\lambda, \tau_1) \prod_{\ell=1}^{n-1} O(\tau_\ell, \tau_{\ell+1}) \cdot [O(\tau_n, \mu) - O(\tau_n, -\mu)] \cdot d^n \tau \\ = \int_0^a [O(\lambda, \tau_1) - O(\lambda, -\tau_1)] \prod_{\ell=1}^{n-1} [O(\tau_\ell, \tau_{\ell+1}) - O(\tau_\ell, -\tau_{\ell+1})] \cdot [O(\tau_n, \mu) - O(\tau_n, -\mu)] \cdot d^n \tau . \end{aligned} \quad (2.14)$$

For $n = 1$, one has that

$$\int_{-a}^a O(\lambda, \tau) [O(\tau, \mu) - O(\tau, -\mu)] \cdot d\tau = \int_0^a O(\lambda, \tau) [O(\tau, \mu) - O(\tau, -\mu)] \cdot d\tau + \int_0^a O(\lambda, -\tau) [O(-\tau, \mu) - O(-\tau, -\mu)] \cdot d\tau \quad (2.15)$$

Above, we have split the integration contour, into the parts $[0; a]$ and $[-a; 0]$ and have changed the variables $\tau \mapsto -\tau'$ in the integral over the segment $[-a; 0]$. It then remains to use the reflection invariance of the kernel $O(-\tau, -\mu) = O(\tau, \mu)$ so as to get the representation for $n = 1$. Assume that (2.14) holds for some n . The ultimate integration over τ_{n+1} occuring in the $n + 1$ -fold integration can be recast in much the same way as for $n = 1$:

$$\int_{-a}^a O(\tau_n, \tau_{n+1}) [O(\tau_{n+1}, \mu) - O(\tau_{n+1}, -\mu)] \cdot d\tau_{n+1} = \int_0^a [O(\tau_n, \tau_{n+1}) - O(\tau_n, -\tau_{n+1})] \cdot [O(\tau_{n+1}, \mu) - O(\tau_{n+1}, -\mu)] \cdot d\tau_{n+1} .$$

Once this operation is performed, the integration in respect to τ_1, \dots, τ_n is of the form taken care of by the induction hypothesis. The result is proven for $n + 1$.

It then follows that

$$R_O(\lambda, \mu) - R_O(\lambda, -\mu) = O(\lambda, \mu) - O(\lambda, -\mu) + \sum_{n \geq 1} \int_{-a}^a [O(\lambda, \tau_1) - O(\lambda, -\tau_1)] \cdot \prod_{\ell=1}^{n-1} [O(\tau_\ell, \tau_{\ell+1}) - O(\tau_\ell, -\tau_{\ell+1})] \cdot [O(\tau_n, \mu) - O(\tau_n, -\mu)] \cdot d^n \tau . \quad (2.16)$$

Since every term in this series is positive, the claim follows. ■

2.3 Existence and uniqueness of the dressed energy

In the Bethe Ansatz litterature, the dressed energy refers to the unique couple $(\varepsilon_0(\lambda \mid q), q)$ solving the conditions

$$\varepsilon_0(\lambda \mid q) - \int_{-q}^q K(\lambda - \mu) \varepsilon_0(\mu \mid q) \cdot \frac{d\mu}{2\pi} = \lambda^2 - h \quad \text{where } q \text{ is such that } \varepsilon_0(q \mid q) = 0 . \quad (2.17)$$

Lemma 2.2 ensures the existence and uniqueness of solutions to all integral equations of the type

$$(I - K/2\pi) \cdot g = f , \quad \text{for any } f \in L^1([- \alpha ; \alpha]) . \quad (2.18)$$

However, the unique solvability of (2.17) is a more involved issue in as much as the endpoint of integration q is also subject to a non-linear equation. In fact the statement about the existence of a solution to the system (2.17) has been made at several places in the litterature [1, 2, 5, 8, 10] although, to the best of the author's knowledge, no proof has ever been given. In the proposition below we prove the unique solvability of the system (2.17).

Proposition 2.1 *Let $\varepsilon_0(\lambda \mid \alpha)$ denote the unique solution to the linear integral equation*

$$\varepsilon_0(\lambda \mid \alpha) - \int_{-\alpha}^{\alpha} K(\lambda - \mu) \varepsilon_0(\mu \mid \alpha) \cdot \frac{d\mu}{2\pi} = \lambda^2 - h . \quad (2.19)$$

Given any α , the function $\lambda \mapsto \varepsilon_0(\lambda \mid \alpha)$ is analytic on the strip $\mathcal{S}_{c/2}$.

Also, there exists a unique $q \in]0; +\infty[$ such that $\varepsilon_0(q \mid q) = 0$. Moreover, there exists open neighborhoods O_q of q and V_0 of 0 in \mathbb{R} such that the function $\alpha \mapsto \varepsilon_0(\alpha \mid \alpha)$ is a C^∞ -diffeomorphism from O_q onto V_0 .

Proof —

Let $R^{(\alpha)}(\lambda, \mu)$ be the resolvent kernel associated with $I - K/2\pi$ understood as acting on $L^2([- \alpha ; \alpha])$. By lemma 2.2, it is an analytic function of $(\lambda, \mu) \in \mathcal{S}_{c/2} \times \mathcal{S}_{c/2}$. Since

$$\varepsilon_0(\lambda \mid \alpha) = \lambda^2 - h + \int_{-\alpha}^{\alpha} R^{(\alpha)}(\lambda, \mu) \cdot (\mu^2 - h) \cdot \frac{d\mu}{2\pi}, \quad (2.20)$$

The statement about the analyticity of $\lambda \mapsto \varepsilon_0(\lambda \mid \alpha)$ on $\mathcal{S}_{c/2}$ follows. Moreover, since $(\alpha, \lambda, \mu) \mapsto R^{(\alpha)}(\lambda, \mu)$ is C^1 and the integration in (2.20) goes through the compact interval $[- \alpha ; \alpha]$, it follows by differentiation under the integral theorems that $\varepsilon_0(\lambda \mid \alpha)$ is continuously differentiable in respect to α .

In order to prove the existence and uniqueness of q , we study the variations of the function $\alpha \mapsto \varepsilon_0(\alpha \mid \alpha)$. For this, we first obtain a representation for its derivative:

$$\frac{d}{d\alpha} \varepsilon_0(\alpha \mid \alpha) = \frac{\partial}{\partial \lambda} \varepsilon_0(\lambda \mid \alpha) \big|_{\lambda=\alpha} + \frac{\partial}{\partial \alpha_0} \varepsilon_0(\lambda \mid \alpha_0) \big|_{\alpha_0=\alpha}. \quad (2.21)$$

It is readily seen through integrations by parts that the last two derivatives satisfy to the integral equations

$$\left[\left(I - \frac{K}{2\pi} \right) \cdot \partial_{\alpha} \varepsilon_0 \right] (\lambda \mid \alpha) = \frac{\varepsilon_0(\alpha, \alpha)}{2\pi} [K(\lambda - \alpha) + K(\lambda + \alpha)] \quad (2.22)$$

$$\left[\left(I - \frac{K}{2\pi} \right) \cdot \partial_{\lambda} \varepsilon_0 \right] (\lambda \mid \alpha) = 2\lambda - \frac{\varepsilon_0(\alpha, \alpha)}{2\pi} [K(\lambda - \alpha) - K(\lambda + \alpha)]. \quad (2.23)$$

These are readily solved in terms of the resolvent kernel

$$\frac{\partial}{\partial \alpha} \varepsilon_0(\lambda, \alpha) = \frac{\varepsilon_0(\alpha \mid \alpha)}{2\pi} [R^{(\alpha)}(\lambda, \alpha) + R^{(\alpha)}(\lambda, -\alpha)] \quad (2.24)$$

$$\frac{\partial}{\partial \lambda} \varepsilon_0(\lambda, \alpha) = -\frac{\varepsilon_0(\alpha \mid \alpha)}{2\pi} [R^{(\alpha)}(\lambda, \alpha) - R^{(\alpha)}(\lambda, -\alpha)] + 2\lambda + \int_{-\alpha}^{\alpha} R^{(\alpha)}(\lambda, \mu) \frac{2\mu}{2\pi} d\mu. \quad (2.25)$$

As a consequence,

$$\frac{d}{d\alpha} \varepsilon_0(\alpha \mid \alpha) = \frac{\varepsilon_0(\alpha \mid \alpha)}{\pi} R^{(\alpha)}(\alpha, -\alpha) + 2\alpha + \int_0^{\alpha} [R^{(\alpha)}(\lambda, \mu) - R^{(\alpha)}(\lambda, -\mu)] \mu \cdot \frac{d\mu}{\pi} \quad (2.26)$$

It follows from lemma 2.3 that the integrand of the integral appearing in the *rhs* is strictly positive on $]0; +\infty[$. It follows from (2.7) that $R^{(\alpha)}(\lambda, \mu) > 0$. Thus $\frac{d}{d\alpha} \varepsilon_0(\alpha \mid \alpha) > 0$ as soon as $\varepsilon_0(\alpha \mid \alpha) \geq 0$.

Suppose that there exist a $q > 0$ such that $\varepsilon_0(q, q) = 0$. It then follows from (2.26) that $\alpha \mapsto \varepsilon_0(\alpha \mid \alpha)$ is strictly increasing in some open neighborhood $]q - \eta; q + M[$ of q with $\eta > 0, M > 0$. Let M correspond to the largest real with this property and suppose that $M \neq +\infty$. One has that $\varepsilon_0(\alpha \mid \alpha)$ is strictly increasing on $]q; q + M[$. Hence, by continuity $\varepsilon_0(M + q \mid M + q) > 0$. Thus, one has that

$$\frac{d}{d\alpha} \varepsilon_0(\alpha \mid \alpha) \big|_{\alpha=q+M} > 0, \quad (2.27)$$

contradicting the definition of M . It thus follows that $\varepsilon_0(\alpha \mid \alpha) > 0$ on $]q; +\infty[$. This reasoning shows that $\alpha \mapsto \varepsilon_0(\alpha \mid \alpha)$ has at most one zero on $[0; +\infty]$. It thus remains to show that it has at least one. It follows from

(2.17) that $\varepsilon_0(0 | 0) = -h < 0$. It thus remains to show that $\varepsilon_0(\alpha | \alpha) > 0$ for α large enough. For this, recast (2.20) as

$$\varepsilon_0(\alpha | \alpha) = \alpha^2 - h + \int_{\sqrt{h}}^{\alpha} [R^{(\alpha)}(\alpha, \mu) + R^{(\alpha)}(\alpha, -\mu)](\mu^2 - h) d\mu - \int_0^{\sqrt{h}} [R^{(\alpha)}(\alpha, \mu) + R^{(\alpha)}(\alpha, -\mu)](h - \lambda^2) d\mu. \quad (2.28)$$

Clearly, the first two terms are strictly positive and behave at least as α^2 when $\alpha \rightarrow \infty$. It follows from the bounds obtained in lemma 2.2 that the last term is, at most a $O(\alpha)$. Thence, $\varepsilon_0(\alpha | \alpha) \xrightarrow{\alpha \rightarrow +\infty} +\infty$. The function $\alpha \mapsto \varepsilon_0(\alpha, \alpha)$ being continuous, it admits at least one zero on $]0; +\infty[$ in virtue of the intermediate value theorem.

Finally, since $\alpha \mapsto \varepsilon(\alpha | \alpha)$ is smooth and $[d\varepsilon(\alpha | \alpha)/d\alpha]_{\alpha=q} > 0$, by the local inversion theorem for smooth functions, there exists open neighborhoods O_q of q and V_0 of 0 in \mathbb{R} such that $\varepsilon(\alpha | \alpha)$ is a C^∞ -diffeomorphism from O_q onto V_0 . ■

3 The low-temperature expansions

In this section, building on the results of the previous two sections, we prove theorem 0.1. We start by proving the existence of the asymptotic expansion at order 0, what represents the hardest part of the theorem. Then, by using a convenient rewriting of Yang-Yang's equation (0.2), we show that the knowledge of the existence of an asymptotic expansion up to order $2\ell + 2$ as given in (0.3) guarantees the validity of this asymptotic expansion (0.3) to the order $2\ell + 4$.

3.1 Asymptotic expansion to order 0

Theorem 1.1 guarantees that the unique solution to the Yang-Yang equation is entire and has a unique zero \widehat{q} which, in virtue of proposition 1.1, satisfies $\widehat{q} \geq \sqrt{h}$ uniformly in T . Since $\varepsilon'(\lambda) > 2\lambda$ for $\lambda > 0$, it follows that $|\varepsilon'(\widehat{q})| > \sqrt{h}$ uniformly in T . As a consequence, by the Bloch-Landau theorem, there exists an open neighborhood

$$U_{\widehat{q}} \subset \mathcal{D}_{\widehat{q}, \sqrt{h}/2} \quad \text{and a radius} \quad \delta \geq \frac{\sqrt{h}|\varepsilon'(\widehat{q})|}{22} > \frac{h}{22} \quad \text{such that} \quad \varepsilon : U_{\widehat{q}} \mapsto \mathcal{D}_{0,\delta} \quad (3.1)$$

is a biholomorphism. Above, $\mathcal{D}_{a,\eta}$ stands for the open disk of radius η centered at a . In the following, we shall write ε^{-1} for the associated local inverse of ε , ie $\varepsilon^{-1}(0) = \widehat{q}$. This being established, we continue by decomposing

$$-T \int_{\mathbb{R}} K(\lambda - \mu) \ln \left[1 + e^{-\frac{\varepsilon(\mu)}{T}} \right] \cdot \frac{d\mu}{2\pi} = \int_{-\widehat{q}}^{\widehat{q}} \varepsilon(\mu) K(\lambda - \mu) \frac{d\mu}{2\pi} + \mathcal{V}^{(0)}(\lambda) + \mathcal{V}^{(\infty)}(\lambda).$$

where

$$\mathcal{V}^{(\infty)}(\lambda) = -T \int_{J^\delta} K(\lambda - \mu) \ln \left[1 + e^{-\frac{|\varepsilon(\mu)|}{T}} \right] \cdot \frac{d\mu}{2\pi} \quad \text{and} \quad \mathcal{V}^{(0)}(\lambda) = -T \int_{J_q^\delta \cup J_{-q}^\delta} K(\lambda - \mu) \ln \left[1 + e^{-\frac{|\varepsilon(\mu)|}{T}} \right] \cdot \frac{d\mu}{2\pi}. \quad (3.2)$$

There, we have defined $J_q^\delta = \varepsilon^{-1}(]-\delta; \delta[)$, $J_{-q}^\delta = -\varepsilon^{-1}(]-\delta; -\delta[)$ and we agree upon $J^\delta = \mathbb{R} \setminus \{J_q^\delta \cup J_{-q}^\delta\}$. Note that both J_q^δ and J_{-q}^δ have a positive orientation.

Since ε is even and strictly increasing on $]0; +\infty[$, it follows that $|\varepsilon(\lambda)| \geq \delta$ for any $\lambda \in J^\delta$. As a consequence, for any $\lambda \in \mathbb{R}$, one has

$$|\mathcal{V}^{(\infty)}(\lambda)| \leq T \ln \left[1 + e^{-\frac{\delta}{T}} \right] \times \int_{\mathbb{R}} K(\lambda - \mu) \cdot \frac{d\mu}{2\pi} \leq T \ln \left[1 + e^{-\frac{\delta}{T}} \right]. \quad (3.3)$$

Thus $\mathcal{V}^{(\infty)}(\lambda)$ only generates exponentially small corrections in T . The evenness of ε allows us to bring the integration in $\mathcal{V}^{(0)}$ solely onto J_q^δ . Then the change of variables $\mu = \varepsilon^{-1}(s)$ leads to

$$\mathcal{V}^{(0)}(\lambda) = -T \int_{-\delta}^{\delta} \left\{ \frac{K(\lambda - \varepsilon^{-1}(s)) + K(\lambda + \varepsilon^{-1}(s))}{\varepsilon' \circ \varepsilon^{-1}(s)} \right\} \ln \left[1 + e^{-\frac{|s|}{T}} \right] \cdot \frac{ds}{2\pi}. \quad (3.4)$$

Since

$$\inf_{s \in]-\delta; \delta[} |\varepsilon' \circ \varepsilon^{-1}(s)| \geq \inf_{s \in]\widehat{q} - \frac{\sqrt{h}}{2}; \widehat{q} + \frac{\sqrt{h}}{2}[} |\varepsilon'(s)| \geq \inf_{s \in]\frac{\sqrt{h}}{2}; +\infty[} |\varepsilon'(s)| \geq \sqrt{h}, \quad (3.5)$$

it follows that, for any $\lambda \in \mathbb{R}$,

$$|\mathcal{V}^{(0)}(\lambda)| \leq 4T^2 \frac{\|K\|_{L^\infty(\mathbb{R})}}{2\pi \sqrt{h}} \int_0^{+\infty} \ln \left[1 + e^{-s} \right] ds = T^2 \pi \frac{\|K\|_{L^\infty(\mathbb{R})}}{6 \sqrt{h}}. \quad (3.6)$$

Hence, the Yang-Yang equation can be recast in the form

$$\varepsilon(\lambda) = \lambda^2 - h + \int_{-\widehat{q}}^{\widehat{q}} K(\lambda - \mu) \varepsilon(\mu) \cdot \frac{d\mu}{2\pi} + \mathcal{V}^{(0)}(\lambda) + \mathcal{V}^{(\infty)}(\lambda) \quad (3.7)$$

with the remainders $\mathcal{V}^{(0)}(\lambda) + \mathcal{V}^{(\infty)}(\lambda) = O(T^2)$ being understood in the $L^\infty(\mathbb{R})$ sense. By proposition 1.1 there exists constants w and T_0 such that $\widehat{q} \leq 2\sqrt{w}$ uniformly in $T \in [0; T_0]$. It thus follows from lemma 2.2 that $I - K/2\pi$ understood as an operator on $L^2([-\widehat{q}; \widehat{q}])$ is invertible with inverse $I + R^{(\widehat{q})}/2\pi$. Moreover, as \widehat{q} is bounded for $T \in [0; T_0]$, it follows from (2.8) that the resolvent kernel $R^{(\widehat{q})}(\lambda, \mu)$ is bounded on \mathbb{R}^2 by a T independent constant.

Thus recalling that $\varepsilon_0(\lambda | \widehat{q})$ refers to the unique solution to equation (2.19) (cf proposition 2.1), it follows that

$$\varepsilon(\lambda) = \varepsilon_0(\lambda | \widehat{q}) + \mathcal{V}^{(0)}(\lambda) + \mathcal{V}^{(\infty)}(\lambda). \quad (3.8)$$

The last two function are given by

$$\mathcal{V}^{(u)}(\lambda) = \mathcal{V}^{(u)}(\lambda) - \int_{-\widehat{q}}^{\widehat{q}} R^{(\widehat{q})}(\lambda, \mu) \mathcal{V}^{(u)}(\mu) \quad \text{with either} \quad u = 0 \text{ or } u = \infty. \quad (3.9)$$

As the resolvent kernel $R^{(\widehat{q})}(\lambda, \mu)$ is bounded on \mathbb{R}^2 by a T independent constant, it follows that

$$\|\mathcal{V}^{(0)}\|_{L^\infty(\mathbb{R})} = O(T^2) \quad \text{whereas} \quad \|\mathcal{V}^{(\infty)}\|_{L^\infty(\mathbb{R})} = O(e^{-\frac{\delta}{T}}) = O(T^\infty). \quad (3.10)$$

This means that the zero \widehat{q} of $\varepsilon(\lambda)$ solves the equation

$$\varepsilon(\widehat{q}) = \varepsilon_0(\widehat{q} | \widehat{q}) + O(T^2) = 0. \quad (3.11)$$

It has been show in proposition 2.1 that there exists a unique $q \in]0; +\infty[$ such that $\varepsilon_0(q | q) = 0$. Moreover, there also exists open neighborhoods O_q of q and V_0 of 0 in \mathbb{R} such that $m : \alpha \mapsto \varepsilon_0(\alpha | \alpha)$ is a C^∞ -diffeomorphism from O_q onto V_0 . Since for T small enough, one has $O(T^2) \in V_0$, it follows that $\widehat{q} = m^{-1}(-O(T^2))$ is well defined. Moreover, the Taylor expansion of m^{-1} around 0 implies that $\widehat{q} = q + O(T^2)$.

Note that one has the decompositon

$$\int_{-\widehat{q}}^{\widehat{q}} \varepsilon(\mu) K(\lambda - \mu) \cdot \frac{d\mu}{2\pi} = \int_{-q}^q \varepsilon(\mu) K(\lambda - \mu) \cdot \frac{d\mu}{2\pi} + \int_q^{\widehat{q}} \varepsilon(\mu) [K(\lambda - \mu) + K(\lambda + \mu)] \cdot \frac{d\mu}{2\pi}. \quad (3.12)$$

We have already established that $\varepsilon(\lambda) = \varepsilon_0(\lambda | \widehat{q}) + O(T^2)$ with the $O(T^2)$ being uniform in $\lambda \in \mathbb{R}$. Since $\varepsilon_0(\lambda | \widehat{q})$ is uniformly bounded in some neighborhood containing q and \widehat{q} , knowing that $\widehat{q} = q + O(T^2)$ ensures that the second term in (3.12) is a $O(T^2)$, again in the $L^\infty(\mathbb{R})$ sense. This means that, in fact, $\varepsilon(\lambda)$ satisfies to

$$\varepsilon(\lambda) = \lambda^2 - h + \int_{-q}^q K(\lambda - \mu) \varepsilon(\mu) \frac{d\mu}{2\pi} + \widetilde{\mathcal{V}}^{(0)}(\lambda) + \mathcal{V}^{(\infty)}(\lambda), \quad (3.13)$$

where

$$\widetilde{\mathcal{V}}^{(0)}(\lambda) = \mathcal{V}^{(0)}(\lambda) + \int_q^{\widehat{q}} \varepsilon(\mu) [K(\lambda - \mu) + K(\lambda + \mu)] \cdot \frac{d\mu}{2\pi}. \quad (3.14)$$

Again, $\widetilde{\mathcal{V}}^{(0)}(\lambda) + \mathcal{V}^{(\infty)}(\lambda) = O(T^2)$ in the $L^\infty(\mathbb{R})$ sense. Thus, by acting with the operator $I + R^{(q)}/2\pi$ on both sides of (3.13) and using, again, the estimates (2.8) we get that $\varepsilon(\lambda) = \varepsilon_0(\lambda | q) + O(T^2)$. The analytic properties of the resolvent kernel $R^{(q)}(\lambda, \mu)$ obtained in proposition 2.1 guarantee that, given any $a \in]0; c/2[$, $\varepsilon_0(\lambda | q)$ is indeed analytic on the strip S_a . As ε is entire, it thus follows that the reminder has also to be holomorphic on S_a . Finally, it is readily seen from its explicit expression that the reminder is stable under finite order λ -differentiations. This last result can be as well derived from Cauchy's integral representation using that the remainder is holomorphic and uniform.

3.2 Existence to all orders

We now establish the existence of the asymptotic expansion to all orders in ℓ by induction. Thus assume that equation (0.3) given in theorem 0.1 holds for some ℓ .

Observe that, in some vicinity of 0, the local inverse $\varepsilon^{-1}(\lambda)$ of $\varepsilon(\lambda)$ admits the integral representation

$$\varepsilon^{-1}(\lambda) = \oint_{\partial U_{\widehat{q}}} \frac{z \varepsilon'(z)}{\varepsilon(z) - \lambda} \frac{dz}{2i\pi} \quad \text{for} \quad \lambda \in \mathcal{D}_{0, \delta/2}. \quad (3.15)$$

For any $z \in \partial U_{\widehat{q}}$ one has $\varepsilon(z) \in \partial \mathcal{D}_{0, \delta}$. Thus, for any $\lambda \in \mathcal{D}_{0, \delta/2}$ one has $|\varepsilon(z) - \lambda| \geq \delta/2$ so that the integral representation (3.15) is indeed well defined. Moreover, this representation is well fit for providing a low- T asymptotic

expansion for the function ε^{-1} in the neighborhood of 0 as soon as the one for ε is known. In particular, it implies that ε^{-1} admits a low- T asymptotic expansion to the same order as ε .

So as to push the asymptotic expansion of $\varepsilon(\lambda)$ two orders further, we first recast equation (3.13) in the form

$$\varepsilon(\lambda) = \varepsilon_0(\lambda | q) + I^{(1)}(\lambda) + I^{(2)}(\lambda) + I^{(\infty)}(\lambda), \quad (3.16)$$

where

$$I^{(1)}(\lambda) = -T \int_{-\delta}^{\delta} \left\{ \frac{\bar{R}(\lambda, \varepsilon^{-1}(s))}{\varepsilon' \circ \varepsilon^{-1}(s)} \right\} \ln \left[1 + e^{-\frac{|s|}{T}} \right] \cdot \frac{ds}{2\pi} \quad \text{where} \quad \bar{R}(\lambda, \mu) = R^{(q)}(\lambda, \mu) + R^{(q)}(\lambda, -\mu), \quad (3.17)$$

and

$$I^{(2)}(\lambda) = \int_{\widehat{q}} \varepsilon(\mu) \cdot \bar{R}(\lambda, \mu) \cdot \frac{d\mu}{2\pi} \quad I^{(\infty)}(\lambda) = -T \int_{\delta} \bar{R}(\lambda, \mu) \ln \left[1 + e^{-\frac{|\varepsilon(\mu)|}{T}} \right] \cdot \frac{d\mu}{2\pi}. \quad (3.18)$$

Moreover, below, we fix a particular value of a , $0 < a < c/2$ defining the width of the strip \mathcal{S}_a . First, we obtain the asymptotic expansion of $I^{(1)}(\lambda)$. It follows from the Taylor integral expansion

$$\frac{\bar{R}(\lambda, \varepsilon^{-1}(s))}{\varepsilon' \circ \varepsilon^{-1}(s)} = \sum_{r=0}^{2\ell+1} \frac{s^r}{r!} \frac{\partial^r}{\partial s^r} \left\{ \frac{\bar{R}(\lambda, \varepsilon^{-1}(s))}{\varepsilon' \circ \varepsilon^{-1}(s)} \right\}_{|s=0} + g_{2\ell+2}(\lambda, s)$$

with

$$g_r(\lambda, s) = \frac{s^r}{(r-1)!} \int_0^1 (1-u)^{r-1} \cdot \frac{\partial^r}{\partial s^r} \left\{ \frac{\bar{R}(\lambda, \varepsilon^{-1}(s))}{\varepsilon' \circ \varepsilon^{-1}(s)} \right\}_{|s=tu} du$$

that one has the representation

$$I^{(1)}(\lambda) = - \sum_{r=0}^{\ell} \frac{T^{2(r+1)}}{(2r)!} \frac{\partial^{2r}}{\partial s^{2r}} \left\{ \frac{\bar{R}(\lambda, \varepsilon^{-1}(s))}{\varepsilon' \circ \varepsilon^{-1}(s)} \right\}_{|s=0} \int_{-\frac{\delta}{T}}^{\frac{\delta}{T}} s^{2r} \ln \left[1 + e^{-|s|} \right] \cdot \frac{ds}{2\pi} - T \int_{-\delta}^{\delta} g_{2\ell+2}(\lambda, s) \ln \left[1 + e^{-\frac{|s|}{T}} \right] \cdot \frac{ds}{2\pi}.$$

As the asymptotic expansion of ε is uniform in respect to differentiations and that \bar{R} is smooth with bounded derivatives on the strip $\bar{\mathcal{S}}_a \times \bar{\mathcal{S}}_a$, it follows that

$$|g_{2\ell+2}(\lambda, s)| \leq s^{2\ell+2} C_{\ell} \quad \lambda \in \bar{\mathcal{S}}_a \quad \text{and} \quad s \in [-\delta; \delta] \quad (3.19)$$

where C_{ℓ} is a T independent constant. Thence

$$\left| T \int_{-\delta}^{\delta} g_{2\ell+2}(\lambda, s) \ln \left[1 + e^{-\frac{|s|}{T}} \right] \cdot ds \right| \leq 2C_{\ell} T^{2\ell+4} \int_0^{+\infty} u^{2\ell+2} \ln[1 + e^{-u}] \cdot du = O(T^{2\ell+4}) \quad (3.20)$$

Furthermore, up to exponentially small corrections in T , one can replace the integration over $[-\delta/T; \delta/T]$ with one over \mathbb{R} . The corrections will, again, be analytic, uniform in $\lambda \in \mathcal{S}_a$ and stable under finite order differentiation. Thence, using the integral representation for the Riemann ζ -function

$$(1 - 2^{-1-r})\zeta(r+2) = \frac{1}{\Gamma(r+1)} \int_0^{+\infty} t^r \ln(1 + e^{-t}) dt, \quad (3.21)$$

we get

$$I^{(1)}(\lambda) = - \sum_{r=0}^{\ell} 2T^{2(r+1)}(1 - 2^{-1-2r})\zeta(2+2r) \frac{\partial^{2r}}{\partial s^{2r}} \left\{ \frac{\bar{R}(\lambda, \varepsilon^{-1}(s))}{\varepsilon' \circ \varepsilon^{-1}(s)} \right\}_{s=0} + O(T^{2(\ell+2)}). \quad (3.22)$$

By composition of the Taylor expansion of $\bar{R}(\lambda, \mu)$ in μ belonging to a neighborhood of \widehat{q} , and of the asymptotic expansions of $\varepsilon^{-1}(s)$ and $1/(\varepsilon' \circ \varepsilon^{-1}(s))$ into powers of T one readily gets that there exist functions $r_p^{(1)}(\lambda)$, $p = 1, \dots, \ell$ analytic in the strip \mathcal{S}_a such that

$$I^{(1)}(\lambda) = - \sum_{r=0}^{\ell} T^{2(r+1)} r_p^{(1)}(\lambda) + O(T^{2(\ell+2)}). \quad (3.23)$$

Again, the remainder is stable in respect to finite order differentiations, uniform and analytic in \mathcal{S}_a .

Clearly, $I^{(\infty)}(\lambda) = O(T^{\infty})$, with a remainder that enjoys the usual properties. Thus, it remains to estimate $I^{(2)}(\lambda)$. Inserting the asymptotic expansion of $\varepsilon(\lambda)$ up to $O(T^{2(\ell+1)})$ corrections leads to

$$I^{(2)}(\lambda) = \int_q^{\widehat{q}} \bar{R}(\lambda, \mu) \cdot \varepsilon_0(\mu | q) \frac{d\mu}{2\pi} + \sum_{p=1}^{\ell} T^{2p} \int_q^{\widehat{q}} \bar{R}(\lambda, \mu) \cdot \varepsilon_{2p}(\mu) \frac{d\mu}{2\pi} + O(T^{2\ell+4}) \quad (3.24)$$

We were able to replace the uniform $O(T^{2\ell+2})$ appearing in the asymptotic expansion of ε by a $O(T^{2\ell+4})$ since it has been integrated over an interval of length $|\widehat{q} - q| = O(T^2)$. This $O(T^{2\ell+4})$ is differentiation stable, uniform and holomorphic in \mathcal{S}_a . Since $\widehat{q} = \varepsilon^{-1}(0)$, \widehat{q} admits a low- T asymptotic expansion to the same order as ε^{-1} . The regularity of the integrands then ensures that there exist analytic functions $r_1^{(2)}(\lambda), \dots, r_{\ell+1}^{(2)}(\lambda)$ on \mathcal{S}_a such that

$$\sum_{p=1}^{\ell} T^{2p} \int_q^{\widehat{q}} \bar{R}(\lambda, \mu) \cdot \varepsilon_{2p}(\mu) \frac{d\mu}{2\pi} = \sum_{p=1}^{\ell} T^{2p} r_p^{(2)}(\lambda) + O(T^{2\ell+4}). \quad (3.25)$$

The remainder in (3.25) is also readily checked to have the desired properties. Moreover, we do stress that one has gained a supplementary precision T^2 on the remainder since on the one hand the sum starts with a T^2 and on the other hand, one integrates over an interval of length $|\widehat{q} - q| = O(T^2)$.

The first integral in (3.24) needs however a separate attention. Taylor expanding the integral to the order $(\widehat{q} - q)^{\ell+2}$ leads to

$$\int_q^{\widehat{q}} \bar{R}(\lambda, \mu) \cdot \varepsilon_0(\mu | q) \cdot \frac{d\mu}{2\pi} = \sum_{r=0}^{\ell+1} \frac{(\widehat{q} - q)^{r+1}}{(r+1)!} \cdot \frac{\partial^r}{\partial \mu^r} \left\{ \bar{R}(\lambda, \mu) \varepsilon_0(\mu | q) \right\}_{\mu=q} + O((\widehat{q} - q)^{\ell+2}). \quad (3.26)$$

Note that the $r = 0$ terms of this expansion is zero since $\varepsilon_0(q | q) = 0$. Thus, this expansion only contains terms $(\widehat{q} - q)^{r+1}$ with $r \geq 1$. Furthermore taking into account that $\widehat{q} - q = O(T^2)$ and that \widehat{q} admits an asymptotic expansion to order $O(T^{2\ell+2})$, it follows that $(\widehat{q} - q)^{r+1}$ with $r \geq 1$ admits an asymptotic expansion up to $O(T^{2\ell+4})$, ie with a T^2 better precision than \widehat{q} . Again, the regularity properties of the integrand then guarantee that there exist holomorphic functions $r_2^{(3)}(\lambda), \dots, r_{\ell+1}^{(3)}(\lambda)$ on \mathcal{S}_a such that

$$\int_q^{\widehat{q}} \bar{R}(\lambda, \mu) \cdot \varepsilon_0(\mu | q) \cdot \frac{d\mu}{2\pi} = \sum_{p=2}^{\ell+1} T^{2p} r_p^{(3)}(\lambda) + O(T^{2(\ell+2)}). \quad (3.27)$$

Adding up together all of the asymptotic expansions that we have obtained leads to

$$\varepsilon(\lambda) = \varepsilon_0(\lambda | q) + \sum_{p=1}^{\ell+1} \left(r_p^{(1)}(\lambda) + r_p^{(2)}(\lambda) + r_p^{(3)}(\lambda) \right) \cdot T^{2p} + O\left(T^{2(\ell+2)}\right). \quad (3.28)$$

where we agree upon $r_1^{(3)}(\lambda) = 0$. Hence, as claimed, $\varepsilon(\lambda)$ admits a low- T asymptotic expansion up to the order $O\left(T^{2(\ell+2)}\right)$ with coefficients being analytic functions of λ on \mathcal{S}_a . Moreover, clearly, the remainder enjoys the claimed properties. ■

Conclusion

We have proposed a method allowing one to prove the existence of the low-temperature expansion to all orders in T of the unique solution of the Yang-Yang equation. In the course of the proof, we have also established the unique solvability of the system (2.17) for the couple dressed energy-endpoint of the Fermi zone $(\varepsilon_0(\lambda | q), q)$ characterizing the zero temperature behavior of the non-linear Schrödinger model. Our method mainly utilizes the positivity of Lieb's kernel so that it should provide one with analogous results for similar non-linear integral equations describing thermodynamics of other integrable models. Also, we expect that upon a few modifications, our techniques would allow one to approach rigorously the question of the large-volume behavior of various thermodynamic quantities arising in integrable models, such as, for instance, the $1/L$ corrections to the energies of the ground and low-lying excited states.

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